Lecture 24: Computing eigenvalues and eigenvectors

December 6, 2016 1:00 PM

Recall from last lecture…

$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$; we want to find all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}$, x $\neq 0$, such that $Ax = \lambda x$. In this case:

λ is an **eigenvalue** and x is an **eigenvector**.

$\det(A - \lambda I) \to$ "characteristic polynomial"

$= \begin{vmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{vmatrix}$ $\det(M) = |M|$

$= (3-\lambda)(2-\lambda) - 2$

$= \lambda^2 - 5\lambda + 4$

$= (\lambda - 1)(\lambda - 4)$

$\lambda = 1, \quad \lambda = 4$ ✲eigenvalues

$\lambda = 1$:

$\ker(A - 1 \cdot I) = \cdots = \left\{ s \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| s \in \mathbb{R} \right\} = E_1$ → the eigenspace

Basis: $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$\lambda = 4$

$\ker(A - 4 \cdot I) = \cdots = \left\{ s \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \middle| s \in \mathbb{R} \right\} = E_4$

Basis: $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

Definition

An nxn matrix A is called "**diagonizable**" if there's a basis of $\mathbb{R}^n$ consisting entirely of eigenvectors in A.

$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ is diagonizable because $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is such a basis of $\mathbb{R}^2$ consisting entirely of A's eigenvectors.

Remember: any matrix with determinant $\neq 0$ is invertible

Let P be the matrix whose columns form the basis that consists entirely of eigenvectors.

$P := \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

Note that: $\det(P) = 3$, so $\neq 0$, so P is invertible.

Observe:

$AP = \left[A \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad A \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right] = \left[1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad 4 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]$

$= P \cdot \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$\Rightarrow P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ → the diagonal matrix with the eigenvalues in its diagonal (in the same order as in P)

22.5 Problematic Cases

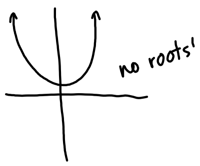
a) Not enough real roots

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\det(A - \lambda I)$

$= \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix}$

$= \lambda^2 + 1$



irrelevant for this course but: you can use complex numbers to solve

b) Not enough eigenvectors

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$\det(A - \lambda I)$

$= \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)(2-\lambda) = (2-\lambda)^2$

only eigenvalue: $\lambda = 2$

$\lambda = 2$

$\ker(A - 2I) = \ker \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$S = \left\{ \begin{pmatrix} s \\ 0 \end{pmatrix} \middle| s \in \mathbb{R} \right\}$

basis: $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

→ we can't find a basis of $\mathbb{R}^2$ consisting entirely of eigenvectors because we only have one vector

23 Final conclusions about diagonizability

The multiplicity of λ as a root of the characteristic polynomial is called the **algebraic multiplicity** of λ. Moreover, $\dim(E_\lambda)$ is called the **geometric multiplicity** of λ.

Theorem

Let λ be an eigenvalue. Then, $1 \leq \underbrace{geom. multiplicity}_{\dim(E_\lambda)} \leq alg. multiplicity.$

Theorem

Eigenvectors corresponding to different eigenvalues are linearly independent.

Corollary

1) If an nxn matrix has n distinct eigenvalues, it is diagonizable.
2) If $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A, then:
   A diagonizable
   <=> the sums of the geom. multiplicities is n
   <=> $\dim\left(E_{\lambda_1}\right) + \cdots + \dim\left(E_{\lambda_n}\right) = n$

What to do to decide about diagonizability?

1) Compute $\det(A - \lambda I)$, find all roots.
   This requires a factorization into linear factors. If that's impossible, A is not diagonizable over the reals.
2) Find a basis for each eigenspace.
3) If there are n vectors in all of the these bases,
   → diagonizable
   If there are fewer than n vectors in all of these bases,
   → not diagonizable
4) In the case of diagonizability, write the basis vectors into the columns of P. Then,
   $P^{-1}AP = D$